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Multivectorial Representation of Lie Groups

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In vector spaces of dimension $n = p+q$ a multivector (Clifford) algebra $\mathscr{C}(p, q)$ can be constructed. In this paper a multivector $\mathscr{C}(p, q)$ representation, not restricted to the bivector subalgebra $\mathscr{C}^2(p, q)$, is developed for some of the Lie groups more frequently used in physics. This representation should be especially useful in the special cases of (grand) unified gauge field theories, where the groups used do not always have a simple tensor representation.

1. INTRODUCTION

Although W. K. Clifford published his first paper (Clifford, 1878) defining the "geometric algebras" in 1878, it is only recently (Chisholm and Common, 1986) that the Clifford algebras have become more than an interesting mathematical possibility and that they are being used to solve some technical problems in mathematics and in physics, inasmuch as they provide a unified universal algebra and mathematical language. This paper shows in particular how representations of Lie groups can be written in that unified structure, avoiding a series of technical difficulties. Furthermore, this construction can become a basic tool for a logical structural model in field theories ranging from grand unified theories to string field models (including supersymmetry) (Ross, 1985). Clifford algebras or multivector algebras can themselves be represented by matrix algebras; then in a sense it is not surprising that multivectors can be used to represent Lie groups which are so often represented by matrices. Here we give a direct method to find the multivector representations.

2. CLIFFORD ALGEBRAS

We define a Clifford algebra as follows (see, for example, Hestenes, 1966; Hestenes and Sobczyk, 1984; Lounesto, 1980).

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Definition 2.1. A clifford algebra $\mathscr{C}(p, q)$ is an associative ring over the real (or complex) field F and simultaneously is a vector space over the same field, such that for all A, B (called multivector) $\in \mathscr{C}(p, q)$ (signature of generators p, q as defined below) and α, $\beta \in F$

$$A = \langle A\rangle_0 + \langle A\rangle_1 + \langle A\rangle_2 + \cdots = \sum_r \langle A\rangle_r; \qquad r = 0, 1, \ldots, p+q \tag{2.1}$$

where $\langle A\rangle_r$ is called the r-vector part of A. If $A = \langle A\rangle_r$ for some positive integer r, then A is said to be homogeneous of grade r and will be called an r-vector. The elements of $\langle A\rangle_r$ have the following properties:

$$\langle A+B\rangle_r = \langle A\rangle_r + \langle B\rangle_r \tag{2.2a}$$

$$\langle \lambda A\rangle_r = \lambda\langle A\rangle_r = \langle A\rangle_r \lambda \qquad \text{if} \quad \lambda = \langle \lambda\rangle_0 \tag{2.2b}$$

Formally,

$$\mathscr{C}(p, q) = \mathscr{C}^0(p, q) + \mathscr{C}^1(p, q) + \cdots + \mathscr{C}^{p+q}(p, q) \tag{2.3}$$

in an obvious decomposition in r-vector parts.

The relation between r-vectors in $\mathscr{C}(p, q)$ is given by their structure and their product in the algebra ($\circ$) thus:

$$\text{if} \quad A = \langle A\rangle_1, \qquad \text{then} \quad A \circ A = A^2 = \langle A^2\rangle_0 + \langle A^2\rangle_2 \tag{2.4a}$$

Furthermore, a member of a linearly independent set of elements of A is called a basis vector e_i of a given representation if in that representation $e_i^2 = \langle A^2\rangle_0$. For $\mathscr{C}(p, q)$ there are p normalized basis vectors with $e_i^2 = +1$ and q normalized basis vectors with $e_i^2 = -1$ (Dauns, 1988):

$$(e_i \circ e_j)_{\text{symmetrized}} = g_{ij} = \text{diag}(1, \ldots 1, -1, \ldots -1) \quad \text{with } p+q = n \text{ elements} \tag{2.4b}$$

We assume that all multivectors can be developed as an r-blade sum (Hestenes and Sobczyk, 1984); an A_r multivector is an r-blade (or a simple r-vector) if and only if it can be factored into a product of r mutually anticommuting vectors $a_1, a_2, \ldots, a_r$, that is,

$$A_r = a_1 a_2 \ldots a_r \tag{2.5}$$

where

$$a_j a_k = -a_k a_j \tag{2.6}$$

for j, $k = 1, 2, \ldots, r$ and $j \neq k$. Also, $r \leq p+q = n$.

A multivector is called even if $A = \langle A\rangle_0 + \sum_n \langle A\rangle_{2n}$, $n = 1, 2, \ldots$.

Finally, for every nonzero r-blade A_r, there exists a nonzero vector a in $\mathscr{A}(p, q) = \mathscr{C}^1(p, q)$ (basis vector) such that $A_r a$ is an $(r+1)$-blade. There are $2^n - 1$ possible r-blades for a set a_i.

Our definition is only one of a series of possible definitions for Clifford algebras (and perhaps it is not the best one for some particular purposes), but it is clear and practical for our presentation and can be used properly to deduce all geometrical operations which are important in a physical model (inner product, outer product, Lie bracket, etc.).

On the other hand, for our definitions we can easily get matrix representations for the multivector algebra using square matrices $m \times m$ [square matrices are required to define the algebra and geometrical products given by (2.1), (2.2), and (2.4)] with a minimum of 2^n degrees of freedom over the real field or 2^{n+1} degrees of freedom over the complex field.

3. MULTIVECTORIAL REPRESENTATION FOR LIE GROUPS

Group theory provides a natural mathematical language for describing symmetries of the physical world; in particular, we have witnessed the increasing application of group theory to physics and to many other scientific areas. In quantum mechanics and quantum field theory in particular it is needed as a powerful tool in exploring both the traditional discrete and continuous space-time symmetries and in elucidating the origin of internal symmetries of nature (gauge invariance) and permutation symmetry (Wu Ki Tung, 1985).

The basic definition of the Clifford algebra presents $\{\mathscr{C}(p, q), +\}$ as an Abelian group (+ being the algebra's sum) and we will construct a multivector representation for Lie groups assigning to every Lie generator a multivector M such that the set $\{M\}$ reproduces the Lie algebra of the group generators.

In short, we will exhibit the isomorphism from an abstract group G to a group of multivector operators $U(G)$ on $\mathscr{C}(p, q)$ (if the representation is faithful, the homomorphism is also an isomorphism, a degenerate representation being one which is not faithful); let us be more specific: we will construct the faithful mapping

$$g \in G \xrightarrow{U} U(g) \tag{3.1}$$

where $U(g)$ is a multivector operator on $\mathscr{C}(p, q)$, such that (Artin, 1957)

$$U(g_1)U(g_2) = U(g_1 g_2) \tag{3.1a}$$

i.e., the multivector operators satisfy the same rules of multiplication as the original group elements.

Following Hestenes and Sobczyk (1984), we start our construction by defining an orthogonal transformation of the set of 1-vectors, remembering that $\{\langle \mathscr{A}(p, q)\rangle_1\} = \mathscr{C}^1(p, q)$:

$$(h(A) \circ h(B))_{\text{symmetrized}} = h(A \circ B)_{\text{symmetrized}} \tag{3.2}$$

for each A, B in $\mathscr{A}(p, q)$ (these are the isometries of the inner product). The group of all orthogonal transformations of $\mathscr{A}(p, q)$ is called the orthogonal group of $\mathscr{A}_{p,q}$ and denoted by $O(p, q)$. A basis vector or a versor v in $\mathscr{C}(p, q)$ is then defined as a multivector that can be factored into a product of k-vectors. A unit versor obeys $(v \circ v)_{\text{symmetrized}} = \pm 1$.

The multiplicative group of all invertible versors in $\mathscr{C}(p, q)$ is the Clifford group $\mathscr{C}(p, q)$ and the multiplicative group of unit versors in $\mathscr{C}(p, q)$ is the versor group $V(p, q)$, which is 2:1 homomorphic (Wu Ki Tung, 1985) to $O(p, q)$; thus, the structure of $O(p, q)$ and its subgroups [think, for example, in $SO(p, q)$] can be described using an algebraic analysis. The multiplicative group of all even unit versors in $\mathscr{C}(p, q)$ is called the spin group of $\mathscr{A}_{p,q}$ [Spin(p, q)] and the rotor group of $\mathscr{A}_{p,q}$, Spin$^\dagger(p, q)$, is the group of all the special versors in $\mathscr{C}(p, q)$ such that $S^\dagger S = 1$ (where $S^\dagger = a_r \cdots a_1$ if $S = a_1 a_2 \cdots a_r$ because $a_i^\dagger = a_i$), called rotors; obviously the rotor group is a subgroup of the spin group.

For the applications to physics we regard, as usual, the spacetime as a pseudo-Euclidean vector space $\mathscr{A}(1, 3)$, the orthogonal group $O(1, 3)$ is the Lorentz group (its elements are called Lorentz transformations), and $SO^\dagger(1, 3)$ is the proper Lorentz group. Spin$^\dagger(1, 3)$ is called the spin-1/2 representation of the proper Lorentz group (Bugajska, 1986a–c).

In this paper, we assume the fundamental theorem of Lie group theory (the generators of a Lie group form a Lie algebra) as true; thus, a classification of Lie groups will be carried out by classifying Lie algebras. In a first type of representation we construct an associative algebra isomorphic to a Lie algebra where the Lie bracket is written as

$$[A, B] = \tfrac{1}{2}(AB - BA) \tag{3.3}$$

with A and $B \in C^2(p, q)$ [bivector algebra of $C(p, q)$]. All subalgebras of $C(p, q)$ closed under (3.3) are Lie algebras.

We are this far following in fact the Hestenes and Sobczyk presentation of this problem, where these authors use the "MIC" hypothesis that every Lie algebra is isomorphic to a bivector algebra, useful to represent the most interesting Lie groups, but in our paper we will show alternative ways to get a multivector representation for some Lie groups and their connection with an associated spinor–multivector system where the MIC idea is not used.

According to the MIC hypothesis, it is possible to construct the Lie algebra of the special unitary group $SU(n)$ and its generalization $SU(p, q)$ as a subalgebra of $B(2p, 2q)$ associated with $\mathscr{A}_{p,q}$ as follows; we select a basis in $\mathscr{A}_{2p,2q}$ of vectors $e_1, \ldots, e_n$ and $f_1 = e_{n+1}, \ldots, f_n = e_{2n}$ with the properties

$$\begin{aligned} e_i \cdot e_j &= f_i \cdot f_j = g_{ij} \\ e_i \cdot f_j &= 0 \end{aligned} \tag{3.4}$$

with $i, j = 1, \ldots, n$, and the "dot" product $a \cdot b = \frac{1}{2}[a \circ b + b \circ a]$. From these vectors we get $n^2 - 1$ linearly independent bivectors [a basis for $B(2p, 2q)$] using the "wedge" product $a \wedge b = \frac{1}{2}[a \circ b - b \circ a]$:

$$\begin{aligned} E_{ij} &= e_1 \wedge e_j + f_i \wedge f_j \\ F_{ij} &= e_i \wedge f_j - f_i \wedge e_j \\ H_k &= e_k \wedge f_k - e_{k+1} \wedge f_{k+1} \end{aligned} \tag{3.5}$$

where $i, j = 1, \ldots, n$ and $k = 1, \ldots, n-1$ with $i \neq j$.

We can now present a multivectorial analysis for typical Lie groups used in modern field theory applying the MIC hypothesis and comparing the result with multivectorial representations using nonbivector subalgebra in an associated spinor-multivector system.

3.1. $SU(0,5)$

This group on a Euclidean vector space $\mathscr{A}(0,5)$ was one of the first used in grand unified theories to model elementary particle interactions; it appears to be a useful step toward the final answer in this kind of theory.

To get a multivectorial representation for $SU(0,5)$ (according to the MIC hypothesis), we construct the Euclidean vector space $\mathscr{A}_{0,10}$ with the basis e_1, e_2, e_3, e_4, e_5, f_1, f_2, f_3, f_4, f_5 and the metric tensor

$$e_i \cdot e_j = f_i \cdot f_j = g_{ij} = \mathrm{diag}(-1, -1, -1, -1, -1) \tag{3.6}$$

As the basis bivectors for $\mathscr{C}^2(0, 10)$, we can use the following nonsimple bivectors:

$$\begin{aligned} E_{12} &= e_1 \wedge e_2 + f_1 \wedge f_2 \qquad & F_{12} &= e_1 \wedge f_2 - f_1 \wedge e_2 \\ E_{13} &= e_1 \wedge e_3 + f_1 \wedge f_3 \qquad & F_{13} &= e_1 \wedge f_3 - f_1 \wedge e_3 \\ E_{14} &= e_1 \wedge e_4 + f_1 \wedge f_4 \qquad & F_{14} &= e_1 \wedge f_4 - f_1 \wedge e_4 \\ E_{15} &= e_1 \wedge e_5 + f_1 \wedge f_5 \qquad & F_{15} &= e_1 \wedge f_5 - f_1 \wedge e_5 \\ E_{23} &= e_2 \wedge e_3 + f_2 \wedge f_3 \qquad & F_{23} &= e_2 \wedge f_3 - f_2 \wedge e_3 \end{aligned}$$

$$
\begin{aligned}
E_{24} &= e_2 \wedge e_4 + f_2 \wedge f_4 & F_{24} &= e_2 \wedge f_4 - f_2 \wedge e_4 \\
E_{25} &= e_2 \wedge e_5 + f_2 \wedge f_5 & F_{25} &= e_2 \wedge f_5 - f_2 \wedge e_5 \\
E_{34} &= e_3 \wedge e_4 + f_3 \wedge f_4 & F_{34} &= e_3 \wedge f_4 - f_3 \wedge e_4 \\
E_{35} &= e_3 \wedge e_5 + f_3 \wedge f_5 & F_{35} &= e_3 \wedge f_5 - f_3 \wedge e_5 \\
E_{45} &= e_5 \wedge e_5 + f_4 \wedge f_5 & F_{45} &= e_4 \wedge f_5 - f_4 \wedge e_5 \\
H_1 &= e_1 \wedge f_1 - e_2 \wedge f_2 & H_3 &= e_3 \wedge f_3 - e_4 \wedge f_4 \\
H_4 &= e_2 \wedge f_2 - e_3 \wedge f_3 & H_4 &= e_4 \wedge f_4 - e_5 \wedge f_5
\end{aligned}
\tag{3.7}
$$

such that they satisfy the Lie algebra for the 24 generators of the $SU(5)$ group (it is a faithful representation). For example,

$$
\begin{aligned}
[E_{ij}, E_{ik}] &= 2E_{jk} & [E_{ij}, F_{ij}] &= -2H_j \\
[E_{ij}, E_{kl}] &= 0 & [H_i, H_j] &= [H_i, H_i] = 0 \\
[F_{ij}, F_{ik}] &= 2E_{jk} & [H_i, E_{ij}] &= -2F_{ij} \\
[F_{ij}, F_{kl}] &= 0 & [H_i, E_{jk}] &= 2F_{jk} \quad \text{with } j = i+1
\end{aligned}
\tag{3.8}
$$

These relations are in $\mathscr{C}(0, 10)$, which then contains the $SU(0, 5)$ group and the $\mathscr{C}(0, 5)$ group. When the vectors used in $\mathscr{C}(0, 5)$ and $\mathscr{C}(0, 10)$ are normalized, the versors of these groups are isomorphic to the $V(0, 5)$ and $V(0, 10)$ groups (we should remind the reader that the versor groups are 2:1 homomorphic to the orthogonal group).

We have presented the multivector representation of $SU(0, 5)$ in this form for physical reasons; first, because in this method the necessity to enlarge the basis space $\mathscr{A}_{0,5}$ to $\mathscr{A}_{0,10}$ appears in a natural way and second because the construction of $\mathscr{C}(0, 10)$ leads to the $O(0, 10)$ group and to the $SO(0, 10)$ group directly. The $SO(0, 10)$ group is very important in the vertical grand unified theories [these theories improve some $SU(0, 5)$ predictions and avoid some problems]; moreover, this construction allows a logical coordination with the exceptional groups [predicting horizontal fundamental interactions between families (Ross, 1985)]. Then, if we use one single mathematical language to both construct Lie groups useful in grand unified theories and to include the spontaneous symmetry breaking (SSB), within the Clifford language, we can get naturalness in the model besides avoiding some representation problems.

Using the spin group definition, the spin(0, 10) group contains the multivectors in $\mathscr{C}(0, 10)$ such that they are invariant if $e_i \to -e_i$ and $f_i \to -f_i$ [the $SU(0, 5)$ group is a subgroup of the spin(0, 10) group]; on the other hand, the elements of the spin(0, 5) group are invariants for the $e_i \to -e_i$ involutions in $\mathscr{C}(0, 10)$ and they form another $SU(0, 10)$ subgroup. It is also possible to show the isomorphism between the spin(0, 5) group and $\mathscr{C}(0,4)$ group. This is a feature of the representations by multivectors which appears in many cases for different (p,q) to a group with $n'=n-1=p+q-1$.

In the $SU(0,5)$ grand unified theory the symmetry is broken to the $SU(3)_c\times SU(2)\times U(1)_y$ model when the scale energy reaches the value predicted from the renormalization group equations; therefore it is useful to show a maximal subgroup $SU(3)_c\times SU(2)\times U(1)_y$ which is contained in $SU(0,5)$ under the following relations:

$$
\begin{aligned}
&E_{12}(SU(0,5))\leftrightarrow E_{12}(SU(0,3))\\
&E_{13}(SU(0,5))\leftrightarrow E_{13}(SU(0,3))\\
&E_{23}(SU(0,5))\leftrightarrow E_{23}(SU(0,3))\\
&F_{12}(SU(0,5))\leftrightarrow F_{12}(SU(0,3))\\
&F_{13}(SU(0,5))\leftrightarrow F_{13}(SU(0,3))\\
&F_{23}(SU(0,5))\leftrightarrow F_{23}(SU(0,3))\\
&H_{1}(SU(0,5))\leftrightarrow H_{1}(SU(0,3))\\
&H_{2}(SU(0,5))\leftrightarrow H_{2}(SU(0,3))\\
&E_{45}(SU(0,5))\leftrightarrow E_{12}(SU(0,2))\\
&F_{45}(SU(0,5))\leftrightarrow F_{12}(SU(0,2))\\
&H_{4}(SU(0,5))\leftrightarrow H_{1}(SU(0,2))\\
&H_{3}(SU(0,5))\leftrightarrow E_{1}(SU(0,1))
\end{aligned}
\tag{3.9}
$$

which are eight $SU(0,3)_c$ generators, three $SU(0,2)$ generators, and one for $U(0,1)$. Thus, the $SU(0,5)\supset SU(0,3)_c\otimes SU(0,2\otimes U(1)_y$ chain is well defined in this multivectorial representation [the other $SU(0,5)$ generators transform under the two non-Abelian groups simultaneously and physically they allow quark-lepton interaction besides the electroweak and strong interactions].

As in this representation we include only Euclidean spaces, the rotor groups [$\text{Spin}^{\dagger}(0,10)$, $\text{Spin}^{\dagger}(0,5)$, and $\text{Spin}^{\dagger}(0,3)$] are isomorphic to the corresponding spin groups [Spin(0, 10), Spin(0, 5), Spin(0, 3)].

3.2. $SU(1,3)$

This is an interesting example because we can use the multivector algebra of the Minkowski space as the algebra for the generator space, and because there are models with grand unified groups larger than $SU(0,5)$

which include it in the intermediate step for the spontaneous symmetry breaking (SSB). In this case we use $\mathscr{A}_{2,6}$ with the basis (e_0, e_1, e_2, e_3; f_0, f_1, f_2, f_3) (according to the MIC hypothesis), such that

$$e_i \cdot e_j = f_i \cdot f_j = g_{ij} = \operatorname{diag}(1, -1, -1, -1) \qquad \text{for} \quad i, j = 0, 1, 2, 3 \quad (3.10)$$

From the basis bivectors for $\mathscr{C}^2(2, 6)$ we can construct the following nonsimple bivectors:

$$\begin{aligned}
E_{01} &= e_0 \wedge e_1 + f_0 \wedge f_1 & F_{01} &= e_0 \wedge f_1 - f_0 \wedge e_1 \\
E_{02} &= e_0 \wedge e_2 + f_0 \wedge f_2 & F_{02} &= e_0 \wedge f_2 - f_0 \wedge e_2 \\
E_{03} &= e_0 \wedge e_3 + f_0 \wedge f_3 & F_{03} &= e_0 \wedge f_3 - f_0 \wedge e_3 \\
E_{12} &= e_1 \wedge e_2 + f_1 \wedge f_2 & F_{12} &= e_1 \wedge f_2 - f_1 \wedge e_2 \\
E_{13} &= e_1 \wedge e_3 + f_1 \wedge f_3 & F_{13} &= e_1 \wedge f_3 - f_1 \wedge e_3 \\
E_{23} &= e_2 \wedge e_3 + f_2 \wedge f_3 & F_{23} &= e_2 \wedge f_3 - f_2 \wedge e_3 \\
H_0 &= e_0 \wedge f_0 - e_1 \wedge f_1 & H_2 &= e_2 \wedge f_2 - e_3 \wedge f_3 \\
H_1 &= e_1 \wedge f_1 - e_2 \wedge f_2
\end{aligned} \qquad (3.11)$$

which satisfy the Lie algebra for the $SU(1, 3)$ group.

It is obvious from the definition of Clifford algebras that the $\mathscr{C}(2, 6)$ group contains the $\mathscr{C}(1, 3)$ group and both are isomorphic to the corresponding versor groups (2:1 homomorphic to their orthogonal groups), and that the Spin(1, 3) group is isomorphic to the $\mathscr{C}(0, 3)$ group, while the $\text{Spin}^{\dagger}(1, 3)$ group (the Lorentz proper group for the spin-1/2 multivector representation) is isomorphic to $\mathscr{C}(0, 2)$. This chain makes $\mathscr{C}(2, 6)$ especially important.

3.3. $SU(0, 3)$

In physics it is of particular interest to analyze the $SU(0, 3)$ group because it is the gauge group used in the quantum chromodynamics theory (QCD); this Yang–Mills field theory is a dynamic principle which explains and predicts some experimental facts of the strong interactions. As a matter of fact, one of the first applications of $SU(0, 3)$ was in the classification of the energy spectrum for hadrons; afterward it was recognized as being even more important in the theory of "color" (Aitchison, 1984).

In the discussion of the $SU(0, 5)$ group above, the $SU(0, 3)$ group was already constructed as a subgroup of $SU(0, 5)$. Here we get a multivector representation for this group enlarging the basis space $\mathscr{A}_{p,q}$ to $\mathscr{A}_{2p,2q}$, a construction which would force the introduction of isotopic spaces and the use of bivector algebras. This would be particularly important in the cases where it is physically necessary to use $\mathscr{A}_{1,3}$ and its Clifford group $\mathscr{C}(1, 3)$.

For the $SU(0,3)$ group it will be shown that it is enough to use a multivector representation of $\mathscr{C}(1,3)$. This Clifford algebra has an irreducible matrix representation over the complex field, 4×4 matrices with 16 linearly independent elements; we can choose these 16 matrices as the four Dirac γ_μ matrices and their products [for groups larger than $SU(0,3)$ it will be necessary to use another matrix set with more degrees of freedom than the Dirac set]. All multivectors M in $\mathscr{C}(1,3)$ can be represented as a linear combination of the Dirac gamma matrices γ_i, in the standard representation, for example. For this reason it is not only possible but also straightforward to find a faithful multivectorial representation for the $SU(0,3)$ group in $\mathscr{C}(1,3)$ and its subgroups. In several cases when the Lie group has 16 generators or less it can have a multivectorial representation in $\mathscr{C}(1,3)$.

In our present case, due to the fact that the $SU(0,3)$ group has only eight generators, it is possible to find several multivectorial representations in $\mathscr{C}(1,3)$ for it. As a first example consider (here $\gamma_{ij\cdots}=\gamma_i\gamma_j\cdots$)

$$\begin{aligned}
\lambda_1&=\frac{1}{2}(\gamma_{01}+i\gamma_{23}) & \lambda_5&=\frac{1}{2}(i\gamma_3-\gamma_{123})\\
\lambda_2&=\frac{1}{2}(\gamma_{02}-i\gamma_{13}) & \lambda_6&=\frac{i}{2}(\gamma_{023}+\gamma_2)\\
\lambda_3&=\frac{1}{2}(\gamma_{03}+i\gamma_{12}) & \lambda_7&=\frac{i}{2}(\gamma_1+\gamma_{013})\\
\lambda_4&=\frac{1}{2}(\gamma_0+i\gamma_{012}) & &\\
\lambda_8&=\left(\frac{i}{\sqrt{3}}\gamma_5+\frac{1}{2\sqrt{3}}\gamma_{03}-\frac{i}{2\sqrt{3}}\gamma_{12}\right) & &
\end{aligned}\tag{3.12}$$

where the λ_i, $i=1,\ldots,8$, satisfy the Lie algebra for $SU(3)$ and

$$\{\gamma_\mu,\gamma_{\mu\nu},\eta_{\mu\nu\rho},\gamma_{\mu\nu\rho\sigma}=\gamma_5\}\in\mathscr{C}(1,3)$$

The λ_1, λ_2, λ_3 (and λ_8) generators constitute a subalgebra included in $\mathscr{C}^2(1,3)$ forming a $\mathscr{C}(1,3)$ multivectorial representation for the $SU(2,0)\otimes U(1,0)$ maximal subgroup [λ_1, λ_2, and λ_3 are generators for the $SU(2,0)$ group and λ_8 is the generator for the $U(1,0)$ group]. The other $SU(0,3)$ generators do not constitute a subalgebra [they are odd versors in $C(1,3)$] and transform under $SU(2,0)$ and $U(0,1)$ simultaneously.

The γ_i $(i=1,2,3)$ generators are invariant under a product with the $i\gamma_5$, while λ_j $(j=4,5)$ are invariant under a product with $i\gamma_{12}$; also the λ_k $(k=6,7)$ are invariant under a product with $\gamma_{03}=-(i\gamma_5)(i\gamma_{12})$; these factors are $U(1)$ generator elements. This observation coincides with concepts presented by Keller (1982, 1984, 1985, 1986, 1991; see also Rodríguez, 1986), who discusses the idempotents which define the projectors needed to classify minimum ideals in $\mathscr{C}(1,3)$ (spinors) and the matrix representations associated with each case; it is convenient to write explicitly

$$\tfrac{1}{2}(\mathbf{1}+i\gamma_5)\lambda_i=\lambda_i, \qquad i=1,2,3, \quad \lambda_i\in SU(2)$$
$$\tfrac{1}{2}(\mathbf{1}+i\gamma_{12})\lambda_j=\lambda_j, \qquad j=4,5$$
$$\tfrac{1}{2}(\mathbf{1}+\gamma_{03})\lambda_k=\lambda_k, \qquad k=6,7$$

for the (3.12) generators, where $\frac{1}{2}(\mathbf{1}+i\gamma_5)$ is the chiral projector and $\frac{1}{2}(\mathbf{1}+i\gamma_{12})$ is the spin projector and the last one is the z-momentum projector. Finally, this multivector structure has a (nontrivial) matrix representation, where it is convenient to use the chiral form for the representation of the Dirac matrices and to choose the chiral and spin projectors to classify minimum ideals in the $\mathscr{C}(1,3)$ (spinors) and conversely this choice generates the chiral representation for the Dirac matrices.

Another construction for $SU(3)$ in $\mathscr{C}(1,3)$ would be given by

$$\lambda'_1=-\frac{i}{2}(\gamma_{23}+\gamma_{023}) \qquad \lambda'_5=\frac{1}{2}(\gamma_3-i\gamma_{123})$$
$$\lambda'_2=\frac{i}{2}(\gamma_{13}+\gamma_{013}) \qquad \lambda'_6=\frac{1}{2}(\gamma_2+i\gamma_{01})$$
$$\lambda'_3=\frac{i}{2}(\gamma_{12}+\gamma_{012}) \qquad \lambda'_7=\frac{1}{2}(\gamma_1-i\gamma_{02})$$
$$\lambda'_4=\frac{1}{2}(\gamma_5+i\gamma_{03}) \qquad \lambda'_8=\frac{1}{2\sqrt{3}}(2\gamma_0+i\gamma_{12}-i\gamma_{012})$$

using the same notation used for the generators (3.12).

These are examples where a subalgebra of $\mathscr{C}(1,3)$ is not its bivector subalgebra. This is needed because the multivector representation for $SU(2)$ has even and odd versors [the even part of $\mathscr{C}(1,3)$ is a subalgebra of it, but the contrary is not necessarily true]; the $SU(2)$ generators are λ'_1, λ'_2, and λ'_3, while the $U(1)$ generator is λ'_8; it is constructed using a lineal combination of multivectors which do not change the group generators [γ_0 for λ'_1, λ'_2, and λ'_3; $i\gamma_{12}$ for λ'_4 and λ'_5; and finally $i\gamma_{012}=(\gamma_0)(i\gamma_{12})$ for λ'_6 and λ'_7]. In this construction

$$\tfrac{1}{2}(\mathbf{1}\pm\gamma_0)\lambda_i=\lambda_i, \qquad i=1,2,3, \qquad \lambda_i\in SU(2)$$
$$\tfrac{1}{2}(\mathbf{1}\pm i\gamma_{12})\lambda_j=\lambda_j, \qquad j=4,5$$
$$\tfrac{1}{2}(\mathbf{1}\pm i\gamma_{012})\lambda_k=\lambda_k, \qquad k=6,7$$

where $\frac{1}{2}(\mathbf{1}\pm\gamma_0)$ is a "mass" projector for the field. This structure has the nontrivial matrix representation induced when we use the mass projector

$[\frac{1}{2}(1 \pm \gamma_0)]$ and spin $[\frac{1}{2}(1 \pm i\gamma_{12})]$ to classify minimum ideals of $\mathscr{C}(1,3)$ (spinors). Here the convenient choice is the standard representation for the Dirac matrices [and their products, see Rodríguez, 1986].

Consequently, we have a general method (deduced from the illustrated with the two special cases presented here) to construct a multivectorial representation for Lie groups. First, we proposed the $SU(3)$ or $SU(2)$ generators using multivectors of interest, either from physical or from mathematical reasons, useful to classify the $\mathscr{C}(1,3)$ minimum ideals (spinors); then, a second set of projectors is chosen in $\mathscr{C}(1,3)$ to construct explicitly the multivector set of a given symmetry (invariant under a set of group generators) and at the end we define the $U(1)$ generator, up to a normalization, as a lineal combination of the three multivectors which leave the group generators invariant (in our case each one of them is the product of the other two). The remaining generators are found as the corresponding product of other previously found generators. Finally, it is important to keep in mind that there is a nontrivial matrix representation for the spinor-multivector system.

3.4. $SU(2)$

Nowadays the most successful gauge model is the $SU(2) \times U(1)$ electroweak interaction model. In it the $SU(2)$ group is locally isomorphic to $SO(3)$ and its square is locally isomorphic to the proper Lorentz group. When we studied the $SU(5)$ group above we got a multivectorial representation to the $SU(3)$ group according to the MIC hypothesis; subsequently, we obtained two multivectorial representations [within $\mathscr{C}(1,3)$] using non-simple multivectors which were not necessarily homogeneous. Due to the fact that $SU(2)$ has only three generators, it has several multivector representations in $\mathscr{C}(1,3)$ [for example, $\{\gamma_{0i}\}$ with $i=1, 2, 3$ or $\{i\gamma_0, i\gamma_0\gamma_5, \gamma_5\}$, which also satisfy the $SU(2)$ Lie algebra].

We will illustrate elsewhere (Keller and Rodríguez-Romo, 1990) the use of this multivector representation for common Lie groups in a spinor-multivector map which will be used to make a multivector analysis of the Dirac equation. Some applications are already included in our previous papers.

In conclusion, it is possible and direct to find multivector representations of Lie groups which are relevant to direct use in the construction of field theories.

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